TECHNICAL NOTE

The Prandtl number effect near the onset of Bénard convection in a porous medium

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This note focuses on Kladias and Prasad's claim that the critical Rayleigh number for the onset of Bénard convection in an infinite horizontal porous layer increases as the Prandtl number decreases, and argues that the critical Rayleigh number (Ra_c) depends only on the Darcy number (Da), as linear stability analysis indicates. The two-dimensional steady-convection problem is then solved numerically to document the convection heat transfer effect of the Rayleigh number, Darcy number, Prandtl number, and porosity. The note concludes with an empirical correlation for the overall Nusselt number, which shows the effect of Prandtl number at above-critical Rayleigh numbers. The correlation is consistent with the corresponding correlation known for Bénard convection in a pure fluid.

Keywords: porous media; Bénard convection; Prandtl number effect

The critical Rayleigh number

Kladias and Prasad (1989) concluded that the critical Rayleigh number for the onset of convection in a horizontal porous layer increases significantly as the Prandtl number becomes sufficiently small. They arrived at this conclusion numerically, by including in their flow model the inertia effect (quadratic, or Forchheimer term) and the no-slip wall effect (Brinkman term), in addition to the usual Darcy-flow term, and by extrapolating the numerical results to the condition of no flow.

The claim that the convection-onset Rayleigh number (Ra_c) increases with the Prandtl number (Pr) was emphasized in a subsequent review article (Prasad and Kladias 1991) on non-Darcy effects in natural convection. For example, Figure 15 in Prasad and Kladias's (1991) review shows the critical Rayleigh number as a function of the Darcy number and Prandtl number. With regard to the Pr effect that is shown clearly in that figure, Prasad and Kladias (1991) wrote that "the onset may be delayed if the Prandtl number is low." In another article (Kladias and Prasad 1991), they wrote that "the Rayleigh number required for the onset of convection increases as the fluid Prandtl number is decreased." This particular claim of Prasad and Kladias is widely known in the field of convection in porous media. It was acknowledged, for example, in the subsequent experimental study conducted by Mouzouris and Burmeister (1990).

It is important to distinguish between the claimed effect of Pr on the critical Rayleigh number, and the Pr effect on Bénard convection, i.e., on fluid motion that *is already present*. The latter was documented in great detail by Kladias and Prasad (1989), and is not questioned in this note. The demonstrated Pr effect in cases where convection is present is compatible with earlier studies, which showed that in natural convection in general the effect of fluid inertia becomes proportionally more important as the Prandtl number becomes small (e.g., Georgiadis and Catton 1986; Plumb and Huenefeld 1981; Wang and Bejan 1987).

The first objective of this note is to point out that, contrary to Kladias and Prasad's (1989) conclusion, the Prandtl number has absolutely no effect on the threshold of convection, i.e., on Ra_c. For this, we draw attention to a short note that was published much earlier by Walker and Homsy (1977) that was apparently overlooked. In that note, the linear stability problem was solved by accounting for the Darcy, Brinkman, and inertial effects in the flow model. Walker and Homsy (1977) showed analytically that if ε is the amplitude of the convective disturbance, then the effect of inertia (taken either as $\mathbf{v}\cdot\nabla\mathbf{v}$ or $\mathbf{v}|\mathbf{v}|$ in the momentum equation) decreases as ε^2, while the Darcy and Brinkman effects decrease as ε. In this way, at the onset of convection (neutral stability), the effect of inertia (Prandtl number) vanishes from the momentum equation. The same conclusion was stressed by Homsy and Sherwood (1976) in a stability study of the horizontal porous layer with through flow.

A perfect analogy exists between Kladias and Prasad's (1989) porous-media claim that Ra_c increases as Pr decreases and the earlier claim made with regard to the onset of convection in pure fluids (Chao et al. 1982; Bertin and Ozoe 1986). In fact, this analogy was offered by Prasad and Kladias (1991) in support of their claim. The earlier conclusion that in pure fluids Ra_c depends on Pr_f was refuted by Proctor (1983), Lage et al. (1991), and Georgiadis (1991): the critical Rayleigh number for a pure fluid is independent of the Prandtl number, $Ra_c = 1707.8$. Lage et al. (1991) showed that Chao et al.'s (1982) conclusion that Ra_c increases as Pr_f decreases is due to the way in which the numerical data for postcritical convection were extrapolated to the base of no convection. The claim made by Kladias and

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Prasad (1989) for porous media is flawed for a similar reason (extrapolation of numerical data for convection), even though the extrapolation curve is not the same as in the pure fluid studies (Chao et al. 1982; Bertin and Ozoe 1986).

Nusselt number correlation

The second objective of this note is defined by the following question: If the Prandtl number has no effect on Ra_c, what effect does it have at Rayleigh numbers immediately above Ra_c? It turns out that near the onset of convection, the calculation of the conduction-referenced overall Nusselt number (Nu_b) requires a high degree of accuracy, because at low Prandtl numbers Nu_b is only slightly greater than 1. We decided to conduct our own series of numerical experiments, because it was not possible to reuse the graphic information reported by Kladias and Prasad (1989). The numerical method and accuracy tests were the same as those described in Lage et al. (1991), and will not be repeated here.

Steady-state convection in a two-dimensional (2-D) infinite porous layer is described by the following mass, momentum, and energy conservation equations:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{1}$$

$$\frac{1}{\phi}(\mathbf{V}\cdot\nabla)U = -\frac{\partial P}{\partial X} + \left(\frac{\mathrm{Pr}}{\mathrm{Ra}}\right)^{1/2}\cdot\nabla^2 U - \frac{F\phi}{\mathrm{Da}}|\mathbf{V}|U - \frac{\phi}{\mathrm{Da}}\left(\frac{\mathrm{Pr}}{\mathrm{Ra}}\right)^{1/2} U \tag{2}$$

$$\frac{1}{\phi}(\mathbf{V}\cdot\nabla)V = -\frac{\partial P}{\partial Y} + \left(\frac{\mathrm{Pr}}{\mathrm{Ra}}\right)^{1/2}\cdot\nabla^2 V - \frac{F\phi}{\mathrm{Da}}|\mathbf{V}|V - \frac{\phi}{\mathrm{Da}}\left(\frac{\mathrm{Pr}}{\mathrm{Ra}}\right)^{1/2} V + \phi\theta \tag{3}$$

$$(\mathbf{V}\cdot\nabla)\theta = \frac{1}{(\mathrm{RaPr})^{1/2}}\nabla^2\theta \tag{4}$$

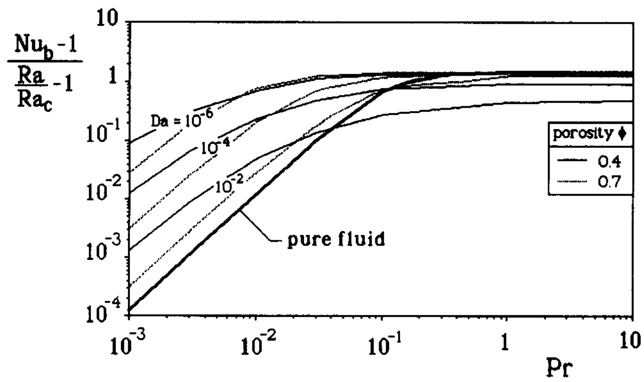

Figure 1 The effect of porosity on the overall Nusselt number near the onset of convection

where $\mathbf{V} = U\mathbf{i} + V\mathbf{j}$ and $\nabla = \partial/\partial X\mathbf{i} + \partial/\partial Y\mathbf{j}$. The nondimensional variables are defined by

$$(X, Y) = \frac{(x, y)}{H}, \quad (U, V) = \frac{(u, v)}{(\alpha_m/H)(\mathrm{RaPr})^{1/2}} \tag{5}$$

$$P = \frac{\phi H^2(p + \rho_f g y)}{\rho_f \alpha_m^2 \mathrm{RaPr}}, \quad \theta = \frac{T - (T_h + T_c)/2}{T_h - T_c}, \quad F = \frac{b}{H} \tag{6}$$

$$\mathrm{Da} = \frac{K}{H^2}, \quad \mathrm{Pr} = \frac{\nu}{\alpha_m}, \quad \mathrm{Ra} = \frac{g\beta H^3(T_h - T_c)}{\alpha_m \nu} \tag{7}$$

while the dimensional (physical variables) are listed in the Notation. The governing equations (Equations 1–4) are based on the homogeneous porous-medium model, with local thermal equilibrium between the solid matrix and the fluid, and the traditional Oberbeck–Boussinesq approximation is made. Finally, in accordance with Kladias and Prasad's (1989) study, the Brinkman and Forchheimer terms are included in the momentum equations (Equations 2 and 3). The boundary conditions that must be satisfied by the solution to Equations 1–4 are as follows: heating from below, $\theta = 1/2$ at $Y = 0$ and $\theta = -1/2$ at $Y = 1$; no slip, $U = V = 0$ at $Y = 0$ and $Y = 1$; and symmetry, $U = 0, \partial(V, \theta)/\partial X = 0$ at $X = 0$ and $X = L/H$.

Notation

b	Inertia coefficient, m
C	Function, Equation 9
$C_{1,2}$	Functions, Equations 13 and 14
Da	Darcy number (K/H^2)
F	Dimensionless inertia coefficient (b/H)
g	Gravitational acceleration, m/s^2
H	Height, m
k_m	Thermal conductivity of saturated porous medium, $W/m\cdot K$
K	Permeability, m^2
Nu_b	Overall bottom-wall Nusselt number, Equation 8
p	Pressure, N/m^2
P	Dimensionless pressure, Equation 6
Pr	Porous-medium Prandtl number (ν/α_m)
Pr_f	Fluid Prandtl number (ν/α_f)
q''_{avg}	Bottom-wall averaged heat flux, W/m^2
Ra	Rayleigh number, Equation 7
Ra_c	Critical Rayleigh number
T	Temperature, K
T_c	Cold wall temperature, K
T_h	Hot wall temperature, K
u, v	Horizontal and vertical volume-averaged velocity components, m/s
U, V	Dimensionless horizontal and vertical volume-averaged velocity components, Equation 5
x, y	Horizontal and vertical coordinates, m
X, Y	Dimensionless horizontal and vertical coordinates, Equation 5

Greek symbols

α_f	Fluid thermal diffusivity, m^2/s
α_m	Porous-medium thermal diffusivity $[k_m/(\rho c_P)_f]$, m^2/s
β	Coefficient of volumetric thermal expansion, K^{-1}
θ	Dimensionless temperature, Equation 6
ν	Fluid kinematic viscosity, m^2/s
ρ_f	Fluid density, kg/m^3
ϕ	Porosity

It is assumed that $L = H$: the square domain represents the cross section of one of the 2-D rolls that form at the onset of convection (Georgiadis and Catton 1986).

The convergence of the numerical solution was monitored globally by calculating the overall Nusselt number based on the heat flux averaged over the bottom wall,

$$\mathrm{Nu_b} = \frac{q''_{avg} H}{k_m(T_h - T_c)} = -\int_0^1 \left(\frac{\partial\theta}{\partial Y}\right)_{Y=0} dX \tag{8}$$

To document the effect of the Prandtl number on convection immediately above $\mathrm{Ra_c}$, the porosity was first fixed at 0.4. The inertia coefficient was evaluated based on the Ergun (1952) model

$$F = C(\phi)\mathrm{Da}^{1/2} \tag{9}$$

in which $C(0.4) = 0.55$. The Darcy number was varied in the range $10^{-6} < \mathrm{Da} < 10^{-2}$. In the limiting case represented by the pure fluid, the porosity was assigned the value 1, and the Darcy number was set equal to a sufficiently large number. Numerical tests showed that $\mathrm{Nu_b}$ is no longer sensitive to Da when the Da value exceeds 10^{-2}. In the pure-fluid limit, the porous-medium Prandtl number $\mathrm{Pr} = \nu/\alpha_m$ is the same as the fluid Prandtl number $\mathrm{Pr_f} = \nu/\alpha_f$. The range covered by the porous-medium Prandtl number in the present study was $10^{-3} < \mathrm{Pr} < 100$. The lowest Pr value is one order of magnitude smaller than the smallest value used by Kladias and Prasad (1989).

Table 1 summarizes the cases studied numerically and the $\mathrm{Nu_b}$ values reached in the steady state for $\phi = 0.4$. A similar table was developed for $\phi = 0.7$. Listed at the bottom of each Da column is the theoretical critical Rayleigh number from the linear stability theory for a porous medium (Walker and Homsy 1977). The lowest Ra value reached for each pair (Pr, Da) was limited by the computational time that was required to achieve steady state. This limitation was particularly severe in the low Prandtl number range $\mathrm{Pr} < 0.1$.

A proper correlation of the $\mathrm{Nu_b}$ (Ra, Da, Pr) data accumulated during this study (e.g., Table 1) must account for the correct value of the critical Rayleigh number at the onset of convection ($\mathrm{Ra} = \mathrm{Ra_c}$, $\mathrm{Nu_b} = 1$). The task of finding this correlation is made difficult by the fact that the $\mathrm{Ra_c}$(Da) curve produced by Walker and Homsy's (1977) analysis was reported only in graphic form: this makes its reading approximate. We note in passing that an analytical $\mathrm{Ra_c}$(Da) expression was reported by Poulikakos (1986) for a different configuration (horizontal walls with zero shear). Poulikakos's $\mathrm{Ra_c}$(Da) function has the same qualitative characteristics as the curve reported graphically by Walker and Homsy (1977). To aid in the development of our $\mathrm{Nu_b}$ correlation, we propose the following expression as an approximate analytical substitute for the $\mathrm{Ra_c}$(Da) curve drawn by Walker and Homsy (1977):

$$\mathrm{Ra_c} = \left[\left(\frac{4\pi^2}{\mathrm{Da}}\right)^n + 1707.8^n\right]^{1/n} \tag{10}$$

The best value of the exponent is $n = 0.87$, so that Equation 10 approximates within 1 percent the $\mathrm{Ra_c}$ values read off Walker and Homsy's figure. This error is comparable with the error associated with reading the figure with a ruler and a pencil. The analytical form of Equation 10 is Churchill and Usagi's (1972) contribution, which has been extensively used to correlate data that span the domain between two known asymptotes. In the present case, the asymptotes are the Darcy limit $\mathrm{Ra_c} = 4\pi^2/\mathrm{Da}$, and the pure fluid limit $\mathrm{Ra_c} = 1707.8$. The $\mathrm{Ra_c}$ values calculated with Equation 10 are listed in the bottom line of Table 1.

The effects of Da and, especially, Pr on the overall Nusselt number become easy to correlate if we plot the $\mathrm{Nu_b}$ data in the frame of Figure 1. The data correspond to the cases that are closest to the onset of convection, with $\mathrm{Nu_b}$ values just greater than 1.000000, as noted in Table 1. The dimensionless group used on the ordinate is suggested by Schlüter et al.'s (1965) finite-amplitude perturbation analysis for a pure fluid, which showed that close to the onset of convection (2-D rolls)

Table 1 Summary of steady-state results for the bottom wall-averaged Nusselt number ($\phi = 0.4$)

	$\mathrm{Da} = 10^{-6}$		$\mathrm{Da} = 10^{-4}$		$\mathrm{Da} = 10^{-2}$		Pure fluid	
Pr	Ra	$\mathrm{Nu_b}$	Ra	$\mathrm{Nu_b}$	Ra	$\mathrm{Nu_b}$	Ra	$\mathrm{Nu_b}$
1	10^8	2.620107	10^6	2.373706	4×10^4	2.334216	2.3×10^3	1.395814
	8×10^7	2.253693	8×10^5	2.054819	2×10^4	1.778201	2.1×10^3	1.287064
	6×10^7	1.765388	6×10^5	1.483107	10^4	1.230347	1.9×10^3	1.154012
	4×10^7	1.022294[a]	4×10^5	1.014072[a]	8×10^3	1.127161[a]	1.8×10^3	1.077998[a]
0.1	10^8	2.508642	10^6	1.893652	4×10^4	1.631833	2.3×10^3	1.289227
	8×10^7	2.164503	8×10^5	1.645560	2×10^4	1.362837	2.1×10^3	1.206273
	6×10^7	1.703891	6×10^5	1.400497	10^4	1.100249	1.9×10^3	1.105807
	4×10^7	1.020498[a]	4×10^5	1.011665[a]	8×10^3	1.079548[a]	1.8×10^3	1.047201[a]
0.01	10^8	1.952951	10^6	1.319919	4×10^4	1.230856	2.7×10^3	1.030209
	8×10^7	1.697036	8×10^5	1.228761	2×10^4	1.097080	2.5×10^3	1.020762
	6×10^7	1.364812	6×10^5	1.122868[a]	10^4	1.026792	2.3×10^3	1.009672
	4×10^7	1.010629[a]	4×10^5	1.000000	8×10^3	1.014516[a]	2.2×10^3	1.003370[a]
0.001	10^8	1.125390	10^6	1.018303	4×10^4	1.008477	3.1×10^3	1.000946
	8×10^7	1.093236	8×10^5	1.012572	2×10^4	1.003467	3.0×10^3	1.000634
	6×10^7	1.047579	6×10^5	1.006536[a]	10^4	1.000613[a]	2.95×10^3	1.000470
	4×10^7	1.001386[a]	4×10^5	1.000000	8×10^3	1.000000	2.9×10^3	1.000300[a]
	$\mathrm{Ra_c} = 3.94 \times 10^7$		$\mathrm{Ra_c} = 3.94 \times 10^5$		$\mathrm{Ra_c} = 6.20 \times 10^3$		$\mathrm{Ra_c} \cong 1.7078 \times 10^3$	

[a] Cases closest to the onset of convection ($\mathrm{Nu_b}$ just greater than 1.000000).

the overall Nusselt number behaves as

$$\frac{\mathrm{Nu_b} - 1}{\frac{\mathrm{Ra}}{\mathrm{Ra_c}} - 1} = (0.69942 - 0.00472\ \mathrm{Pr_f^{-1}} + 0.00832\ \mathrm{Pr_f^{-2}})^{-1} \quad (11)$$

Figure 6 in Lage et al. (1991) showed that the numerical results produced by the present method agree with Equation 11 in the pure-fluid limit.

Guided by Equation 11 and Figure 1, for a horizontal porous layer saturated with fluid we propose an empirical correlation of the form

$$\frac{\mathrm{Nu_b} - 1}{\frac{\mathrm{Ra}}{\mathrm{Ra_c}} - 1} = [(C_1\ \mathrm{Pr}^2)^{-m} + C_2^{-m}]^{-1/m} \quad (12)$$

in which C_1, C_2, and m are at the most functions of Darcy number and porosity. The $\mathrm{Ra_c(Da)}$ function is provided by Equation 10. In order to determine C_2 accurately, the numerical calculations were extended to Pr values as high as 100. The best combinations (C_1, C_2, m) are correlated quite accurately (within 2 percent) by

$$\phi = 0.4: C_1 \cong 172\ \mathrm{Da}^{-0.516}, C_2 \cong 0.295\ \mathrm{Da}^{-0.121}, m \cong 0.4 \quad (13)$$

$$\phi = 0.7: C_1 \cong 30\ \mathrm{Da}^{-0.501}, C_2 \cong 1.21\ \mathrm{Da}^{-0.013}, m \cong 0.7 \quad (14)$$

In the pure-fluid limit ($\phi = 1$), the role of correlation is played by Equation 11. It is worth noting that the right side of Equation 12 approximates the right side of Equation 11 with an accuracy better than 0.6 percent if $C_1 \cong 120$, $C_2 \cong 1.43$, and $m \cong 1.05$. It is a coincidence that in the three porosity cases that we have considered ($\phi = 0.4, 0.7, 1$), the optimal value of the exponent m is nearly equal to the value of the porosity ϕ. The curves of Figure 1 were plotted using the proposed correlation (Equations 12–14).

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